Muses on the Gregorian calendar

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Pope Gregory XIII and the Gregorian calendar

Friday 7 January 1502 – Saturday 10 April 1585[1]

The Gregorian calendar in use today is not so old. The United Kingdom, its colonies, including the American colonies, adopted the reforms in September of 1752 during the reign of King George II. Before this, the realm was using the Julian calendar, and this had accounted fairly well for the Earth's last quarter turn in its annual orbit. However, after one and one half millennia, the more accurate fraction of 0.24219 was beginning to reveal itself in the form of a concerning week and a half retreat of the vernal equinox.

The simplicity of the Julian calendar is perhaps why it lasted for so long—one leap year every four years—but the effect of the 0.00781 difference was first noticed by the venerable Bede as early as AD 725, who noted that the full moon was ahead of its tabulated date. Some 777 years later, in 1582, Pope Gregory XIII signed a papal bull heralding a new and improved calendar for the world. While most Catholic countries adopted the changes immediately, the protestant countries, included the United Kingdom opted for deferral. It took another 170 years before the UK, on 22 May 1751, enacted the Gregorian calendar into law for itself and her colonies.

The changes made in the United Kingdom were predominantly threefold. Strange as it may seem, New Years Day, under the old arrangements, was not 1 January of each year. Rather, it was celebrated on the 25th day of March each year, the day known to Catholics as the Feast of the Annunciation and colloquially as Lady's Day. The new arrangements made 31 December 1751 the last day of that year, shortening it to a mere 282 days. The New Year, as for all future new years would then begin on 1 January as is still the custom today. However, that change did not fix the slip in the vernal equinox. The new law also determined that the day following Wednesday 2 September 1752 would be Thursday 14 September 1752, thereby shortening that year by 11 days. The new arrangements also put in place a mechanism that ensured that

1 The day of the week that Pope Gregory XIII was born was determined using the Julian calendar.

there would be virtually no more slippage of the seasons in future centuries. All end of centuries (1800, 1900, 2000, etc.) would not be leap years unless they were divisible by 400.

Christian Zeller

Monday 24 June 1822 – Wednesday 31 May 1899

Turning our discussion to the calendar itself, we find that the way in which months are assigned a certain number of days is essentially a remnant of history. The variations in the number of days in each month made it difficult to establish a simple mathematical formula for the pattern. However, on 16 March 1883, a school inspector in Germany by the name of Christian Zeller announced at a meeting of the Mathematical Society of France that he had stumbled on a function that could be used to determine the number of the elapsed days of a year up to and including a specified date.

To understand how his function works, we need to think of each month's total days as made up of 28 days plus an excess that we can call x_n where the subscript n is the month number (e.g., $n = 1$ for January, $n = 2$ for February).

Zeller showed that the total excess of days that accrue from the *beginning of March* to just prior to the *beginning of month n* is given by:

$$Z_n = \sum x_n = \left\lfloor \frac{13[\mathrm{mod}(n+9,12)]+2}{5} \right\rfloor \tag{1}$$

The formula uses the floor function $\lfloor x \rfloor$ which, for non-negative numbers, takes the integer part of the number x only. Readers might also be unfamiliar with the modular expression $[\mathrm{mod}(n+9,\ 12)]$ in the numerator. Think of this as the left-overs when $n + 9$ is divided by 12. So for example, if $n = 6$, the left-overs when 15 is divided by 12 is 3.

Continuing, if we let $n = 6$ (the number for June, the sixth month) into equation (1), then $Z_6 = 8$ showing that there are 8 excess days in the completed months March, April and May (March and May have three of these and May has two).

$$Z_6 = \sum x_6 = \left\lfloor \frac{13[3]+2}{5} \right\rfloor = 8$$

Again suppose we wish to know the total excess of days from the beginning of March to just prior to the beginning of February (that is, to 31 January). We let $n = 2$ into (1) to find $Z_2 = 29$. The reader can check that this is in fact correct.

We can put Zeller's function to the task of determining the number days that have elapsed in any given common year (a non-leap year) in the following way. Suppose we wish to know the number of days that have elapsed in the year up to and including, say, 21 September.

First note that $Z_9 = 16$ so that there are 16 *excess* days from the beginning of March to the end of August. This means that the total number of days during that time period is $6 \times 28 + 16 = 184$ days. Adding the 59 days of January and February and the 21 days of September to this yields a total elapsed day count of 264.

Can we now construct a formula for $E_{d,n}$, the total number of elapsed days up to and including the day d of the month n in a common (non-leap) year? First consider that for months after February we have $E_{d,n} = Z_n + 28(n-3) + 59 + d$. This accounts for the 59 days of January and February plus the $(n-3)$ packages of 28 plus Z_n and d. With a slight simplification, this becomes $E_{d,n} = Z_n + 28n - 25 + d$.

For January or February, the formula adjusts to $E_{d,n} = Z_n + 28(n+9) + 59 - 365 + d$. To see why, think about the number of elapsed days to, say, 16 February. Here $d = 16$ and $n = 2$. From the beginning of March to the end of January is $9 + 2$ or 11 months. This means that there are $308 + Z_2 + 16$ or 353 days between 1 March and 16 February inclusive. If we add the 59 days of January and February, and then subtract a full 365 days, we are left with 47, the precise number of days from 1 January to 16 February. Again the formula simplifies to $E_{d,n} = Z_n + 28n + d - 54$.

Putting it all together we can state the following. To determine $E_{d,n}$, the total number of days that have elapsed in any *common year* up to and including day d of month n we have the following:

If $Z_n = \left\lfloor \dfrac{13[\operatorname{mod}(n+9,12)]+2}{5} \right\rfloor$

then the total number of days up to and including the dth day of the nth month of a common year is given by:

$$E_{d,n} = \begin{cases} Z_n + 28n + d - 54 & n \le 2 \\ Z_n + 28n + d - 25 & n > 2 \end{cases}$$

NB: For leap years, this formula will need to be adjusted up by 1 for $n > 2$.

Box 1

We could use the equations in Box 1 to determine the day of the week for any particular date of the year, provided we know on which day 1 January falls. For example, suppose we wished to know what day of the week 21 September 2011 falls on. Using $E_{d,n} = Z_n + 28n + d - 25$, we find that $E_{21,9} = Z_9 + 28 \times 9 + 21 - 25 = 264$ as before. Because 1 January was a Saturday, and knowing that $264 \equiv 5 \pmod 7$, we can conclude that 21 September was four days after Saturday, i.e., Wednesday.

Charles Dodgson (aka Lewis Carroll)

Friday 27 January 1832 – Friday 14 January 1898

Charles Dodgson (also known as Lewis Carroll) devised an efficient method for determining the day of the week given any date in the Gregorian calendar. With sufficient practice, the technique becomes an extremely useful one, and can provide a little entertainment value to others. Indeed, legend has it that it was one of Dodgson's favourite party tricks, usually announcing that he could determine the correct day of the week within 20 seconds. The author has learned the technique himself and enjoyed pronouncing to others the day of the week for things like dates of births, future dates for the current year, famous dates in history and other events. The method is simple enough, but does require memorising a few numbers associated with the calendar months. As a trick for students at most year levels, the technique strengthens basic arithmetic skills, involving a little mental computation with multiplication and division.

To use the method, Dodgson first considered that any date could be broken up into four parts. These are:

1. the century part (that is the first two digits of the year),
2. the year part (the last two digits of the year)
3. the month part (January, February, etc.)
4. the day of the month part (the 5th, the 17th, etc.)

So for example, in 21 September 2017, the century part is 20, the year part is 17, the month part is September and the day part is 21. We could imagine a table of parts as shown in Table 1.

Table 1. Illustration of parts of the date.

Parts	Example	Carroll Number
Century	20	c
Year	17	y
Month	September	m
Day	21	d

From here Dodgson determines the numbers c, y, m and d corresponding to each of the four parts as follows:

Century: To find c, determine the remainder when the century part is divided by 4 (remainder 0), then take this remainder away from 3, and finally double the result. For our example, you should find the century part 20 has the Carroll number 6 given as $2 \times (3 - 0)$.

Year: To find y, first determine how many *complete sets of twelve* are in the year part, how many are *left over* after these sets of twelve are removed, and how many *complete sets of four* are in those left-overs. Then add those three numbers up. For our example, the number of twelves in 17 is 1, the left-overs are 5 and there is 1 four in the left-overs—that is a total of 7.

Month: Commit to memory the month Table 2 for m, as shown

Table 2. Value of m for each month.

m	Month
0	January, October
1	May
2	August
3	February, March, November
4	June
5	September, December
6	April, July

Day: The day part d is the day number itself. So for the 21st, $d = 21$. So Table 1 is completed as shown in Table 3.

Table 3. From date parts to Carroll numbers.

Parts	Example	Carroll number
Century	20	$c = 6$
Year	17	$y = 7$
Month	September	$m = 5$
Day	21	$d = 21$

From here, we simply add the Carroll numbers up and find the left-overs when that sum is divided by 7. Using modular arithmetic, we determine $6 + 7 + 5 + 21 = 4(\text{mod } 7)$.

The great news about modular arithmetic is that we can reduce each of the four numbers modulo 7 before we add them. So 7 and 21 in this example add nothing to the remainder and thus $6 + 5 = 11 = 4 \ (\text{mod } 7)$ is slightly easier to work out.

The last hurdle involves converting the final remainder to a day of the week as shown in Table 4:

Table 4. Relationship between final remainder and day of week.

Final remainder	Day
0	Sunday
1	Monday
2	Tuesday
3	Wednesday
4	Thursday
5	Friday
6	Saturday

So 21 September 2017 will be a Thursday.

One slight complication occurs when dealing with leap years. When dealing with January or February of a leap year, subtract one from the final remainder.

For example, the date 21 February 2020 has Carroll numbers 6, 11, 3, and 21 and these sum to 6(mod 7). However, because 2020 is a leap year, the correct day is Friday.

The author has thought of a few tricks to help with the method. First of all, the century number can only be one of four numbers in the ordered sequence 6, 4, 2, 0. The years of the current century all have the Carroll number 6, and better still, the years of the previous century (the 1900s) all have the Carroll number 0. To help with Table 2, April and July are the only months with the letter l, June has four letters, September and December are the 'three e' months, May Day was 1 May, and 3 is a *FeMiNine number* (February, March and November).

A couple of other points can be made. When dealing with remainders the four Carroll numbers can be reduced modulo 7 at any time. For example, the 21 in our example above could have immediately been reduced to 0 (mod 7) and so adds nothing to the final remainder. Also, most of the useful determinations involve the current year, and so we can prepare a corresponding number that combines the first two Carroll numbers.

For example, the first two Carroll numbers associated with 2014 are 6 and 3, so for that year we hold what we might call the *Carroll value* of 9(mod7) = 2 in our heads. Suppose the day of the week for 7 August is required. I simply add 7 + 2 = 9 and then add the 2 (for 2014) to get 11 or in other words 4(mod 7) or Thursday.

We can note other things as well. For example, the calendar for the two months September and December and for the two months April and July must always have their dates and days aligned—the same applies for February, March and November in common years. Also, 1 January and 31 December in any common year have the same Carroll value of 1 (0 + 1 = 5 + 31 (mod 7)). This implies that in any common year, the first and last days of the year must fall on the same day of the week.

Now while the explanation looks lengthy, in practice, days of the week can be determined very quickly, particularly so for a given year. Let me give you a taste of how to *think* using the method. As stated above, 2014 has a combined Carroll Value of 2 (mod 7) so for example the day for 25th March 2014 is 25 (which is really 4) and 3 and 2, or a total of 9, or 2 (mod 7), or Tuesday. Similarly Christmas day is 4 and 5 and 2, or 4 (mod 7), or Thursday. In fact September and December as 5, together with 2 for 2014, means that the date in those months indicates the day. That is, for example, 14 September 2014 is a Sunday and the 1 December 2014 is a Monday.

It is a nice idea to determine, for each month, a Carroll Value which incorporates both the month and year. For example, suppose we are just about to commence the month of March in 2014. The Carroll Value is 3 + 2 = 5 (mod 7). If you commit 5 to memory for a month, then the day for any date in

that month becomes obvious. For example, 16 March is a Sunday (a multiple of 7), the month starts on Saturday, there are five Saturdays for March, etc. In fact, after a while, with practice, knowing the Carroll value for a particular year, and committing to memory the Carroll month numbers (which never change), gives you a really practical tool to use on a day-to-day basis. Charles Dodgson has also provided us with an interesting device that can be applied in the classroom. The possibilities, for example, with young students are endless. Imagine marking the Calendar with the yearly Carroll value, and getting students to commit to memory the Carroll month numbers. Each morning, the day is confirmed with something like, “It’s the 23rd February 2016, so 23 plus 3 plus 5 for the year equals 31 and the left-over after the sevens are cast out is 3, so it would be Wednesday, but this year is a leap year, so its Tuesday!” Perhaps for a slightly older group we hear, “I was born on April Fool’s Day 2003, so 6 for the century, and no 12s in 3, so 3 is left over, and no 4s in 3, so it’s 9 for 2003, which is the same as 2, then the 1st of April is 1 plus 6 which is 7, and with the 2 is 9, which is really 2. I was born on Tuesday!” Teachers could introduce some analysis of Carroll numbers. Take 25th March 1956 for example—interesting because the Carroll value is 0 plus 0 plus 0. How often does that occur? How many times will your birthday fall on a Saturday? How many calendars are possible? The list of questions is endless. At higher year levels, we could explore modular arithmetic elements, and investigate why the Carroll numbers work, and could a similar system be devised for the Julian calendar. At all levels though, learning these things provides a simple and powerful tool for life and strengthens basic mental computation in a fun way.

John Conway on the day of the week

John Conway, Professor of Mathematics at Princeton University, notes the amazing fact that the kth day of the kth month, for $k = 4, 6, 8, 10, 12$ will always fall on the same day irrespective of the year! So also the fifth day of the ninth month and vice versa and the seventh day of the eleventh month and vice versa (easily remembered by the phrase “Working 9 to 5 in a 7/11 store”)—all falling on the same day in the calendar. In fact, he names it the Doomsday for that year. For example, the Doomsday for 2014 is Friday. So 4 April, 6 June, 8 August, 10 October, 12 December, 9 May, 5 September, 7 November and 11 July are all Fridays in 2014.

Conway suggests that knowing the Doomsday for any particular year is a useful key for determining the day of the week. For example, in 2011, the doomsday was Monday. A date such as 21 September 2011 is therefore 16 days ahead of Monday 5 September—Wednesday. Of course, Conway’s rule corresponds to the Carroll numbers as well. The fourth of April has a Carroll value of 3(4 + 6), along with 6 June, 8 August, etc.

The assignment of the days of the week

As a final note, Richards (1998) gives us a fascinating insight into the origins of the sequence of day names in the modern calendar. The week of 7 days, although not officially sanctioned until much later, was becoming popular in the Roman world around the time of the birth of Christ. Each of the seven days was assigned to a 'planet' according to the assumed 'distance' each planet was away from the Earth, and this happens to corresponds to each planet's orbital period as shown in Table 5.

Table 5. Assignment of Planets to days.

Planet	Orbital period	Assigned hours						
Saturn	29 years	1	8	15	22	5	12	19
Jupiter	12 years	2	9	16	23	6	13	20
Mars	687 days	3	10	17	24	7	14	21
Sun	365 days	4	11	18	1	8	15	22
Venus	224 days	5	12	19	2	9	16	23
Mercury	88 days	6	13	20	3	10	17	24
Moon	29 days	7	14	21	4	11	18	1

According to the Roman historian Dion Cassius (AD 150–235), the first hour of the first 24 hour day was assigned the planet Saturn being the slowest of the heavenly wanderers. Each subsequent hour of that first day was assigned the planets in order of presumed distance from the Earth. After the 7th hour (assigned to the Moon), the cycle begins to repeat, over and over again until the 24th hour (assigned to Mars). Then the first hour of the second day is assigned to the very next planet on the list: the Sun. The cycle keeps repeating through to the 1st hour of the 3rd day (assigned to the Moon) as shown in table 4. Progressively, the first hour of each new day is assigned a new planet, and since 7 and 24 are relatively prime, no planet can be assigned to the same day more than once across the 168 hours of a week. Hence we arrive at Table 6:

Table 6. Allocation of day names.

1st hour planet	English day names	French names
Saturn	Saturday	Samedi
Sun	Sunday	Dimanche
Moon	Monday	Lundi
Mars	Tuesday	Mardi
Mercury	Wednesday	Mercredi
Jupiter	Thursday	Jeudi
Venus	Friday	Vendridi

I have included the French naming equivalents in Table 5 to illustrate that we only need two countries modern naming conventions to capture all of the remnants of these planet allocations.

The calendar is a fascinating study of mans' attempt to measure time by the observation of the motions of the Moon around the Earth and the Earth around the Sun. So much of existence—our life cycles, the seasons, our religions, our communications, agriculture and manufacturing, even our military campaigns—depends on measuring time well. An Earth that takes turns per cycle seems designed, and perhaps our adoption of the Gregorian calendar says something about an unspoken acceptance of a dawning truth - the resignation to a world that is not so square.

The algorithms presented in this article could easily be adapted for the classroom. Students could develop computer programs around the Zeller congruencies and junior students in particular could strengthen basic arithmetic concepts by learning the Lewis Carroll technique as well as getting the chance to see mathematics as a powerful and practical pursuit.

References

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